1. a) $y'' - y = xe^x + e^{2x}$

$\lambda^2 - 1 = 0$, $\lambda_{1/2} = \pm 1$

y_H : $y = C_1 e^x + C_2 e^{-x}$

y_P : Ansatz : $(ax+b)xe^x + ce^{2x}$

$y'_P = axe^x + (ax+b)e^x + (ax+b)xe^x + 2ce^{2x}$

$= e^x(2ax + b + ax^2 + bx) + 2ce^{2x}$

$y''_P = (ax^2 + (2a+b)x + b)e^x + (2ax + 2a + b)e^x + 4ce^{2x}$

$(ax^2 + (2a+b)x + b)e^x + (2ax + 2a + b)e^x + 4ce^{2x} - (ax+b)xe^x - ce^{2x}$

$= xe^x + e^{2x}$

$e^x(4ax + 2a + 2b) + 3ce^{2x} = xe^x + e^{2x}$

$4a = 1$, $a = \frac{1}{4}$

$2a + 2b = 0$, $b = -\frac{1}{4}$

$3c = 1$, $c = \frac{1}{3}$

$y = C_1 e^x + C_2 e^{-x} + \frac{1}{4}(x-1)xe^x + \frac{1}{3}e^{2x}$

b) $\frac{y'}{1-2x} = \frac{1}{3y^2}$, $y(3) = 2$

$y' = \frac{1-2x}{3y^2} = \frac{dy}{dx}$

$\int 3y^2\,dy = \int (1-2x)\,dx$

$y^3 = x - x^2 + C$

$y(3) = 2$: $8 = 3 - 9 + C \Rightarrow C = 14$

$y^3 = x - x^2 + 14$ oder $y = \sqrt[3]{x - x^2 + 14}$

c) $y^{(4)} - 8y'' + 7y = 0$

$$\lambda^4 - 8\lambda^2 + 7 = 0$$

$$\lambda^2_{1|2} = 7 \text{ und } 1$$

$$y = C_1 e^x + C_2 e^{-x} + C_3 e^{\sqrt{7}x} + C_4 e^{-\sqrt{7}x}$$

d) $y'' + y = \frac{1}{\cos x}$

$\lambda^2 + 1 = 0$, $\lambda = \pm i$

$y_H = C_1 \cos x + C_2 \sin x$

$g(x) = \sin x$, $g(0) = 0$, $g'(0) = 1$

$$y_P(x) = \int_0^x g(x+0-t)\frac{1}{\cos t}\,dt = \int_0^x \sin(x-t)\frac{1}{\cos t}\,dt$$

$$= \int_0^x (\sin x \cos t - \cos x \sin t)\frac{1}{\cos t}\,dt$$

$$= \int_0^x \sin x\,dt - \cos x \int \frac{\sin t}{\cos t}\,dt$$

$$= \sin x \cdot t\Big|_0^x - \cos x(-\ln|\cos t|)\Big|_0^x$$

$$= x\sin x - \cos x(-\ln|\cos x| + \ln\cos 0)$$

$$= x \sin x + \cos x \cdot \ln|\cos x|$$

$y = y_H + y_P$.

2) a) $\int\limits_{1/4}^{1} x \sin\sqrt{x}\, dx = \int\limits_{1/4}^{1} x\left(\sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{5})^5}{5!}\right)dx$

$$= \int\limits_{1/4}^{1}\left(x^{\frac{3}{2}} - \frac{1}{3!}x^{\frac{5}{2}} + \frac{1}{5!}x^{\frac{7}{2}}\right)dx$$

$$= \frac{2}{5}x^{\frac{5}{2}} - \frac{1}{6}\cdot\frac{2}{7}x^{\frac{7}{2}} + \frac{1}{120}\cdot\frac{2}{9}x^{\frac{9}{2}}\Bigg|_{1/4}^{1}$$

$$= \frac{2}{5} - \frac{1}{21} + \frac{1}{540} - \frac{2}{5}\left(\frac{1}{4}\right)^{5/2} + \frac{1}{21}\left(\frac{1}{4}\right)^{7/2} - \frac{1}{120}\frac{2}{9}\left(\frac{1}{4}\right)^{9/2}$$

$$= 0{,}342101$$

b) $f(x) = \sqrt{1+x}$, $f(0) = 1$

$f'(x) = \frac{1}{2\sqrt{1+x}}$, $f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}\frac{1}{\sqrt{(1+x)^3}}$, $f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}\frac{1}{\sqrt{(1+x)^5}}$, $f'''(0) = \frac{3}{8}$

Polynom: $y = 1 + \frac{1}{2}x - \frac{1}{4}\cdot\frac{x^2}{2} + \frac{3}{8}\cdot\frac{x^3}{6} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

$\sqrt{2} = \sqrt{1+x} \Rightarrow x = 1$

$$y(1) = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} = \frac{23}{16} = 1{,}4375$$

c) $\frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \frac{x^4}{2^4} + \dots = \sum\limits_{n=1}^{\infty}\frac{x^n}{2^n}$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\frac{1}{2^n}}{\frac{1}{2^{n+1}}} = 2$$

Konvergenzintervall: $(-2,2)$

3) $V(r,h) = \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$

a) $V_r = 2\pi r \left(r + \frac{1}{3}h\right)$

$V_h = \frac{1}{3}\pi r^2$

$dV = \left(2\pi r^2 + \frac{2}{3}\pi r h\right) dr + \frac{1}{3}\pi r^2 \, dh$

b) $h = 18\,cm$, $r = 9\,cm$, $dh = 0{,}5$ $\quad dV = 0$

$0 = \left(2\pi 81 + \frac{2}{3}\pi \cdot 18 \cdot 9\right) dr + \frac{1}{3}\pi 81 \cdot 0{,}5$

$162\pi\left(1 + \frac{2}{3}\right) dr = -\frac{1}{3}\pi\, 40{,}5 \Rightarrow dr = \underline{\underline{-0{,}05}}$

4) $f(x,y) = x^2(e^y - 2) - y^2$

$f_x = 2x(e^y - 2)$
$f_y = x^2(e^y) - 2y$

$f_{xx} = 2(e^y - 2)$
$f_{xy} = 2xe^y = f_{yx}$
$f_{yy} = x^2 e^y - 2$

$f_x = 0$

$x = 0$ $\qquad$ $e^y = 2$, $y = \ln 2$

in f_y: $y = 0$
$P_1(0|0)$

in f_y: $x^2(e^{\ln 2}) - 2\ln 2 = 0$
$2x^2 - 2\ln 2 = 0$
$x^2 = \ln 2$
$x = \pm\sqrt{\ln 2} = \pm 0{,}83255$

$P_2(\sqrt{\ln 2} \,|\, \ln 2)$ $\qquad$ $P_3(-\sqrt{\ln 2} \,|\, \ln 2)$

$f_1(0,0)$: $\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$ Maximum $(0|0)$

P_2: $\begin{vmatrix} 0 & 4\sqrt{\ln 2} \\ 4\sqrt{\ln 2} & 2\ln 2 - 2 \end{vmatrix} = -16\ln 2 < 0$ Sattelpunkt

P_3: $\begin{vmatrix} 0 & -4\sqrt{\ln 2} \\ -4\sqrt{\ln 2} & 2\ln 2 - 2 \end{vmatrix} = -16\ln 2 < 0$ ‖

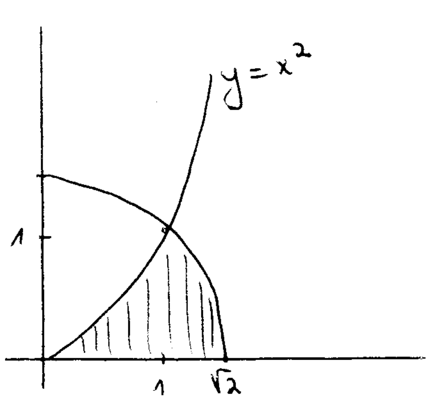
5) $L(x, y, \lambda) = -x - 2y + 20 + \lambda(x^2 + y^2 - 25)$

$L_x = -1 + 2\lambda x = 0 \quad (1)$
$L_y = -2 + 2\lambda y = 0 \quad (2)$
$L_\lambda = x^2 + y^2 - 25 = 0 \quad (3)$

$2 \times (1) - (2)$: $4\lambda x - 2\lambda y = 2\lambda(2x - y) = 0 \Rightarrow y = 2x$
$0 = x^2 + 4x^2 - 25$
$x = \pm\sqrt{5}$

$P_1(\sqrt{5}, 2\sqrt{5})$ $f(P_1) = 20 - 5\sqrt{5}$, Minimum
$P_2(-\sqrt{5}, -2\sqrt{5})$ $f(P_2) = 20 + 5\sqrt{5}$, Maximum

6)



G: $0 \le y \le 1$

$\sqrt{y} \le x \le \sqrt{2 - y^2}$

$\iint_G f\,dG = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{\sqrt{2-y^2}} xy\,dy\,dx$

$= \int_0^1 \frac{1}{2} y (2 - y^2 - y)\,dy = \frac{5}{24}$

7) $$\int_0^2 \int_0^{2\pi} \int_0^{2-r\cos\varphi} 3r^2 \sin^2\varphi \, r \, dr\, d\varphi\, dz$$

$$= \int_0^2 \int_0^{2\pi} 3r^3 \sin^2\varphi \,(2 - r\cos\varphi)\, d\varphi\, dr$$

$$= \int \int (6r^3 \sin^2\varphi - 3r^4 \sin^2\varphi \cos\varphi)\, d\varphi\, dr$$

$$= 3\int_0^2 \left(2r^3 \left[\tfrac{1}{2}\varphi - \tfrac{1}{4}\sin 2\varphi\right]_0^{2\pi} - r^4 \left[\tfrac{1}{3}\sin^3\varphi\right]_0^{2\pi} \right) dr$$

$$= 3\int_0^2 2r^3 \pi \quad = 6\,\frac{r^4}{4}\pi \Big|_0^2 = 24\pi$$

8) $$\frac{x+y}{x^2+y^2} = \frac{r\cos\varphi + r\sin\varphi}{r^2\cos^2\varphi + r^2\sin^2\varphi} = \frac{\cos\varphi + \sin\varphi}{r}$$

$$\int_{r=1}^{4} \int_{\varphi=\frac{\pi}{6}}^{\pi/3} \int_{z=0}^{\frac{\cos\varphi+\sin\varphi}{r}} r\, dz\, d\varphi\, dr = \int_1^4 \int_{\pi/6}^{\pi/3} (\cos\varphi + \sin\varphi)\, d\varphi\, dr$$

$$= \int_1^4 (\sin\varphi - \cos\varphi)\Big|_{\pi/6}^{\pi/3} dr = \int_1^4 \left(\sin\tfrac{\pi}{3} - \cos\tfrac{\pi}{3} - \sin\tfrac{\pi}{6} + \cos\tfrac{\pi}{6}\right) dr$$

$$= \int_1^4 \left(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\right) dr = (\sqrt{3}-1)r\Big|_1^4 = 3(\sqrt{3}-1) \approx 2{,}196$$

9) $$\begin{pmatrix} 1 & 1 & 2 & 6 \\ 2 & -1 & -3 & 4 \\ 4 & 1 & a & 16a \end{pmatrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & -3 & -7 & -8 \\ 0 & -3 & a-8 & 16a-24 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 3 & 7 & 8 \\ 0 & 0 & -1+a & 16a-16 \end{array}\right)$$

$a = +1$: $z = \alpha$

$$3y + 7\alpha = 8 \Rightarrow y = \frac{8 - 7\alpha}{3}$$

$$x = \frac{10}{3} + \frac{1}{3}\alpha$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10/3 \\ 8/3 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} +\frac{1}{3} \\ -\frac{7}{3} \\ 1 \end{pmatrix}$$

$a \neq 1$: $z = 16$, $y = -\frac{104}{3}$, $x = \frac{26}{3}$

10) a)

$$\begin{array}{c|c} & \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} \end{array} = X^T X$$

b)

$$\begin{array}{c|c} & \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} \\ \hline \frac{1}{14}\begin{pmatrix} 5 & -1 \\ -1 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

11)

$$\begin{vmatrix} 3-t & 1 & 1 \\ 2 & 4-t & 2 \\ 1 & 1 & 3-t \end{vmatrix} = \begin{vmatrix} 3-t & 1 & 0 \\ 2 & 4-t & -2+t \\ 1 & 1 & 2-t \end{vmatrix} = \begin{vmatrix} 3-t & 1 & 0 \\ 2 & 4-t & -2+t \\ 3 & 5-t & 0 \end{vmatrix}$$

$$= -(-2+t)\big((3-t)(5-t) - 3\big)$$

$$= (2-t)(12 - 8t + t^2) = (2-t)(t-6)(t-2)$$

$t_1 = 2$, $t_2 = 6$, $t_3 = 2$