1. a) $f(x) = \sum_{n=0}^{\infty} \frac{1}{3^n \cdot (n+1)} \cdot x^n$

$$\lim_{n\to\infty} \left| \frac{\frac{1}{3^n(n+1)}}{\frac{1}{3^{n+1}(n+2)}} \right| = \lim_{n\to\infty} \frac{3^{n+1} \cdot (n+2)}{3^n \ (n+1)} = \lim_{n\to\infty} \frac{3n+6}{n+1} = 3$$

Konvergenzintervall $[-3, 3)$

b) $y = x \ln(x+1)$ $\quad y = 1-2x$

$x \ln(x+1) = 1-2x$ $\quad D_f: \ln(x+1) > 0 \Rightarrow x > -1$

$x\left(x - \frac{x^2}{2} + \frac{x^3}{3} \ldots\right) = 1-2x$

$x^2 - \frac{x^3}{2} + \frac{x^4}{3} = 1-2x$

$x^2 + 2x - 1 = 0$ $\quad x_1 = 0{,}414$

$\quad x_2 = -2{,}414$ außerhalb D_f.

c) $\int_0^1 \frac{1-e^{-x}}{x}\, dx = \int_0^1 \frac{1-\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}\right)}{x}\, dx$

$$= \int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{6}\right) dx = x - \frac{x^2}{4} + \frac{x^3}{18}\Big|_0^1 = 1 - \frac{1}{4} + \frac{1}{18} = \frac{29}{36} = 0{,}8055$$

2. $f(x) = \ln \frac{1}{x}$ $\quad x_0 = 1$

$f(1) = 0$

$f'(x) = \frac{1}{\frac{1}{x}}\left(-\frac{1}{x^2}\right) = -\frac{x}{x^2} = -\frac{1}{x}$ $\quad f'(1) = -1$

$f''(x) = \frac{1}{x^2}$ $\quad f''(1) = 1$

$f'''(x) = -\frac{2}{x^3}$ $\quad f'''(1) = -2$

$$f(x) = 0 + (-1)(x-1) + \frac{1}{2}(x-1)^2 - \frac{2}{6}(x-1)^3$$

$$= -x + 1 + \frac{1}{2}x^2 - x + \frac{1}{2} - \frac{2}{6}\left(x^3 - 3x^2 + 3x - 1\right)$$

$$= -x + 1 + \frac{1}{2}x^2 - x + \frac{1}{2} - \frac{2}{6}x^3 + x^2 - x + \frac{2}{6}$$

$$= \frac{1}{6}\left(-2x^3 + 9x^2 - 18x + 11\right)$$

$\ln 2$? $\quad 2 = \frac{1}{x} \Rightarrow x = \frac{1}{2}$

$$\frac{1}{6}\left(-2\frac{1}{8} + 9 \cdot \frac{1}{4} - 9 + 11\right) = \frac{2}{3}$$

3 gerade Funktion, $b_n = 0$

$$a_0 = \frac{A}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$f(x) = -\frac{2A}{\pi}x + A$ für $x \in [0, \frac{\pi}{2}]$ $\qquad f(x) = 0$ für $x \in [\frac{\pi}{2}, \pi]$

$$a_n = -\frac{4A}{\pi^2}\int_{0}^{\pi/2} x\cos nx\,dx + \frac{2A}{\pi}\int_{0}^{\pi/2}\cos nx\,dx$$

$$= -\frac{4A}{\pi^2}\left(\frac{\pi\sin(n\frac{\pi}{2})}{2n} + \frac{1}{n^2}\cos\left(n\frac{\pi}{2}\right) - \frac{1}{n^2}\right) + \frac{2A}{\pi}\cdot\frac{\sin(n\frac{\pi}{2})}{n}$$

$$= \frac{4A}{\pi^2 n^2}\left(1 - \cos n\frac{\pi}{2}\right) \qquad \text{für } n = 1, 2, 3, \dots$$

$$f(x) = \frac{A}{4} + \frac{4A}{\pi^2}\cos x + \frac{2A}{\pi^2}\cos 2x + \frac{4A}{9\pi^2}\cos 3x + \frac{4A}{25\pi^2}\cos 5x \dots$$

4. a) $y' + 2\frac{y}{x} = e^x$ mit $y(1) = e$

y_H: $y' + 2\frac{y}{x} = 0$

$\frac{dy}{dx} + \frac{2y}{x} = 0$

$\frac{dy}{y} = -\frac{2}{x}dx$

$\ln y = -2\ln x + \ln c$
$= -\ln x^2 + \ln c$
$= \ln \frac{c}{x^2}$

$y = \frac{c}{x^2}$

y_P: $y = \frac{c(x)}{x^2}$

$y' = \frac{c'(x)\cdot x^2 - c(x)2x}{x^4}$

$= \frac{c'(x).x - 2c(x)}{x^3}$

$\frac{c'(x)x - \cancel{2c(x)}}{\cancel{x^3}} + \cancel{\frac{2}{x}\frac{c(x)}{x^2}} = e^x$

$\frac{c'(x)}{x^2} = e^x$

$c'(x) = x^2 e^x$

$c(x) = (x^2 - 2x + 2)e^x$

$y_P = (x^2 - 2x + 2) \cdot \frac{e^x}{x^2}$

$= \left(1 - \frac{2}{x} + \frac{2}{x^2}\right) \cdot e^x$

$y = \frac{c}{x^2} + \left(1 - \frac{2}{x} + \frac{2}{x^2}\right) \cdot e^x$

$y(1) = e$: $e = c + (1 - 2 + 2)e = c + e \Rightarrow c = 0$

$y = \left(1 - \frac{2}{x} + \frac{2}{x^2}\right)e^x$

b) $y + y'\left(x + \frac{1}{y}\right) = 0$

$y + \frac{dy}{dx}\left(x + \frac{1}{y}\right) = 0$

$y\,dx + \left(x + \frac{1}{y}\right)dy = 0$

$f_x = y \Rightarrow f(x,y) = yx + g(y) \Rightarrow f_y(x,y) = x + g'(y)$
$= x + \frac{1}{y}$

$\Rightarrow g'(y) = \frac{1}{y} \Rightarrow g(y) = \ln y + c \Rightarrow \underline{F(x,y) = xy + \ln y + c = (}$

c) $y' = e^{x-y}$

$$y' = \frac{e^x}{e^y}$$

$$e^y\,dy = e^x\,dx$$

$$e^y = e^x + C$$

$$y = \ln(e^x + C)$$

d) $y' \sin x - y = 1 - \cos x$

y_H : $y' \sin x - y = 0$

$$\frac{dy}{y} = \frac{dx}{\sin x}$$

$$\ln|y| = \ln \tan\frac{x}{2} + \ln C = \ln C \tan\frac{x}{2}$$

$$y = C \tan\frac{x}{2}$$

$$y_P = C(x) \tan\frac{x}{2}$$

$$y'_P = C'(x) \tan\frac{x}{2} + C(x) \frac{1}{\cos^2\frac{x}{2}} \cdot \frac{1}{2}$$

$$C'(x) \tan\frac{x}{2} \sin x + C(x) \frac{1}{\cos^2\frac{x}{2}} \cdot \frac{1}{2} \sin x - C(x) \tan\frac{x}{2} = 1 - \cos x$$

$$C'(x) \frac{1-\cos x}{\cancel{\sin x}} \cdot \cancel{\sin x} + \cancel{C(x) \frac{\sin x}{\cancel{2}} \frac{1}{\frac{1}{\cancel{2}}(1+\cos x)}} \cancel{- C(x) \frac{\sin x}{1+\cos x}} = 1 - \cos x$$

$$C'(x)(1-\cos x) = 1 - \cos x$$

$$C'(x) = 1$$

$$C(x) = x$$

$$y_P = x \tan\frac{x}{2}$$

$$y = x \tan\frac{x}{2} + C \tan\frac{x}{2} = (x + C)\tan\frac{x}{2}$$

e) $y'' + 4y' + 4y = \sin x + \sinh 2x + \frac{1}{2}e^{-2x}$

$$= \sin x + \frac{e^{2x}}{2} + \frac{e^{-2x}}{2} - \frac{e^{-2x}}{2}$$

$$= \sin x + \frac{e^{2x}}{2}$$

$y'' + 4y' + 4y = 0$

$\lambda^2 + 4\lambda + 4 = 0 \rightarrow \lambda_{1|2} = -2$

$y_H : C_1 e^{-2x} + C_2 x e^{-2x}$

$y_p : A_1 \sin x + A_2 \cos x + A_3 e^{2x}$

$y'_p : A_1 \cos x - A_2 \sin x + 2A_3 e^{2x}$

$y''_p : -A_1 \sin x - A_2 \cos x + 4A_3 e^{2x}$

$-A_1 \sin x - A_2 \cos x + 4A_3 e^{2x} + 4A_1 \cos x - 4A_2 \sin x + 8A_3 e^{2x}$
$+ 4A_1 \sin x + 4A_2 \cos x + 4A_3 e^{2x} = \sin x + \frac{1}{2} e^{2x}$

$-A_1 - 4A_2 + 4A_1 = 1$

$-A_2 + 4A_1 + 4A_2 = 0$

$4A_3 + 8A_3 + 4A_3 = \frac{1}{2} \Rightarrow A_3 = \frac{1}{32}$

$A_1 = \frac{3}{25}, \quad A_2 = -\frac{4}{25}$

$y_p = \frac{3}{25} \sin x - \frac{4}{25} \cos x + \frac{1}{32} e^{2x}$

$y = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{3}{25} \sin x - \frac{4}{25} \cos x + \frac{1}{32} e^{2x}$

5) $z = (x+y)^2 + e^{2x+y} - \frac{x}{2}\sin y + \frac{3}{4}\cos y$

$z_x = 2(x+y) + 2e^{2x+y} - \frac{1}{2}\sin y$

$z_{xx} = 2 + 4e^{2x+y}$

$z_y = 2(x+y) + e^{2x+y} - \frac{x}{2}\cos y - \frac{3}{4}\sin y$

$z_{yy} = 2 + e^{2x+y} + \frac{x}{2}\sin y - \frac{3}{4}\cos y$

$z_{yx} = 2 + 2e^{2x+y} - \frac{1}{2}\cos y$

$z_{xx} - 3z_{yx} + 2z_{yy}$

$= \cancel{2} + \cancel{4e^{2x+y}} - \cancel{6} - \cancel{6e^{2x+y}} + \frac{3}{\cancel{2}}\cancel{\cos} y + \cancel{4} + \cancel{2e^{2x+y}}$

$+ x\sin y - \frac{3}{\cancel{2}}\cancel{\cos} y$

$= x\sin y$

6) $V(r,h) = \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$

a) $V_r = 2\pi r^2 + \frac{2}{3}\pi r h = 2\pi r\left(r + \frac{1}{3}h\right)$

$V_h = \frac{1}{3}\pi r^2$

$dV = \left(2\pi r^2 + \frac{2}{3}\pi r h\right)dr + \frac{1}{3}\pi r^2 dh$

b) $h = 18\,cm$, $r = 9\,cm$ $dh = 0{,}5$ $dV = 0$

$0 = \left(2\pi \cdot 81 + \frac{2}{3}\pi\ 18 \cdot 9\right)dr + \frac{1}{3}\pi \cdot 81 \cdot 0{,}5$

$162\cancel{\pi}\left(1 + \frac{2}{3}\right)dr = -\frac{1}{3}\cancel{\pi}\ 40{,}5 \Rightarrow dr = \underline{\underline{-0{,}05}}$

7) $z = \ln(xy^2) - \frac{1}{4}xy^2 - 2(x-1)^2 \,, \qquad x>0,\, y\neq 0$

$$z_x = \frac{1}{xy^2}\cdot y^2 - \frac{1}{4}y^2 - 4(x-1) = \frac{1}{x} - \frac{y^2}{4} - 4x + 4 \qquad (1)$$

$$z_y = \frac{1}{xy^2}2xy - \frac{1}{2}xy = \frac{2}{y} - \frac{1}{2}xy \qquad (2)$$

$$z_{xx} = -\frac{1}{x^2} - 4$$

$$z_{xy} = z_{yx} = -\frac{1}{2}y$$

$$z_{yy} = -\frac{2}{y^2} - \frac{1}{2}x$$

$(2)=0$: $\frac{2}{y} = \frac{1}{2}xy \Rightarrow x = \frac{4}{y^2}$ in (1)

$\frac{y^2}{4} - \frac{y^2}{4} - 4\frac{4}{y^2} + 4 = 0 \Rightarrow x = 1,\ y = \pm 2$

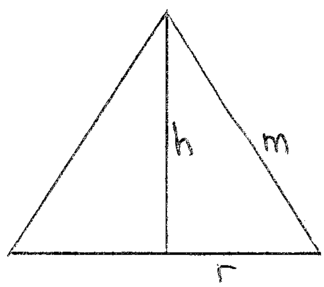
$P_1(1|2) \qquad P_2(1|-2)$

$\text{Hess } f(1|2) = \begin{vmatrix} -5 & -1 \\ -1 & -1 \end{vmatrix} = 4 > 0$ Maximum

$\text{Hess } f(1|-2) = \begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix} = 4 > 0$ "

8) $m = \sqrt{r^2+h^2} = 3 \iff r^2 + h^2 - 9 = 0$

Volumen $V(r,h) = \frac{1}{3}\pi r^2 h$



$$L(r,h,\lambda) = \frac{\pi}{3}r^2 h + \lambda(r^2 + h^2 - 9)$$

$$L_r = \frac{2}{3}\pi r h + 2r\lambda = 0 \quad (1)$$

$$L_h = \frac{1}{3}\pi r^2 + 2\lambda h = 0 \quad (2)$$

$$L_\lambda = r^2 + h^2 - 9 = 0 \quad (3)$$

$$(1)\cdot h - (2) r = \frac{2}{3}\pi r h^2 - \frac{1}{3}\pi r^3 = \frac{1}{3}\pi r(2h^2 - r^2) = 0$$

$$r^2 = 2h^2$$

$$(3):\ 2h^2 + h^2 - 9 = 0$$

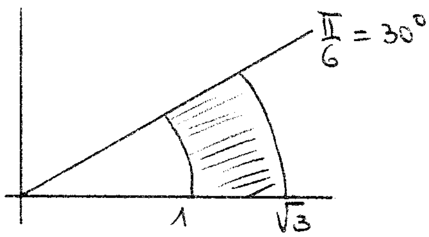
$$h = \sqrt{3}, \quad r = \sqrt{6}$$

9) $$\int_0^4 \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{r\cos\varphi} r \, dr \, d\varphi \, dz$$

$$= \int_0^4 \int_{-\frac{\pi}{2}}^{\pi/2} r^2 \cos\varphi \, d\varphi \, dr = \int_0^4 r^2 \sin\varphi \Big|_{-\frac{\pi}{2}}^{\pi/2} dr = \int_0^4 2r^2 \, dr$$

$$= \frac{2}{3} r^3 \Big|_0^4 = \frac{2}{3}\cdot 64 = \frac{128}{3}$$

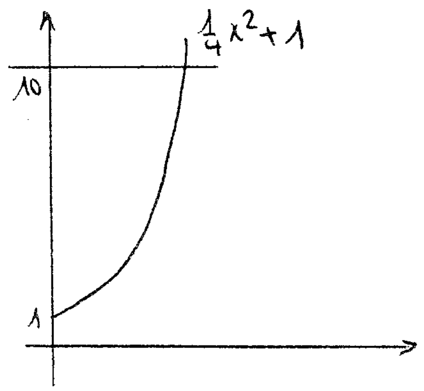
10)



$$M = \int_1^{\sqrt{3}} \int_0^{\pi/6} \frac{4\cancel{r}\cos\varphi}{\cancel{r^2}(1+r^2)} \cancel{r} \, dr \, d\varphi = \int_1^{\sqrt{3}} \frac{4}{1+r^2} \sin\varphi \Big|_0^{\pi/6}$$

$$= \int_1^{\sqrt{3}} \frac{4}{1+r^2} \cdot \frac{1}{2} dr = 2 \arctan r \Big|_1^{\sqrt{3}} = 2(60^\circ - 45^\circ) = \frac{\pi}{6}$$

11)



$$\frac{1}{4}x^2+1=10$$

$$x^2+4=40$$

$$x^2=36$$

$$x=6$$

$$y=\frac{1}{4}x^2+1 \Leftrightarrow y-1=\frac{1}{4}x^2 \Leftrightarrow x^2=4y-4 \Leftrightarrow x=2\sqrt{y-1}$$

$$V=\int_{y=1}^{10}\int_{x=0}^{2\sqrt{y-1}}\int_{z=0}^{x/y} dz\,dx\,dy=\int_1^{10}\int_0^{2\sqrt{y-1}}\frac{x}{y}\,dx\,dy$$

$$=\int_1^{10}\frac{1}{y}\frac{x^2}{2}\Big|_0^{2\sqrt{y-1}}dy=\int_1^{10}\frac{1}{2y}\big(4(y-1)\big)dy=\int_1^{10}\left(2-\frac{2}{y}\right)dy$$

$$=2y-2\ln y\Big|_1^{10}\approx 13{,}395$$

$$V=\int_{x=0}^{6}\int_{y=\frac{1}{4}x^2+1}^{10}\int_{z=0}^{x/y}dz\,dy\,dx \quad \text{ungünstig}$$