Lösungsvorschlag Klausur Mathematik 1 WS 2001/02

Aufgabe 1

a) $3\tan x = 5\cot x$

$$3\tan x = \frac{5}{\tan x}$$

$$3\tan^2 x = 5$$

$$\tan^2 x = \frac{5}{3}$$

$$\tan x = \pm\sqrt{\frac{5}{3}}$$

$x_1 = 52{,}24°;\quad x_2 = 127{,}76°;\quad x_3 = 232{,}24°;\quad x_4 = 307{,}76°$

b) $\frac{1}{2}\lg x = -\lg 8 + \lg\left(\frac{1}{5}x - 1\right) + 1$

$$\lg\sqrt{x} + \lg 8 = \lg\left(\frac{1}{5}x - 1\right) + \lg 10$$

$$\lg 8\sqrt{x} = \lg 10\left(\frac{1}{5}x - 1\right)$$

$$8\sqrt{x} = 10\cdot\left(\frac{1}{5}x - 1\right) = 2x - 10$$

$$64x = 4x^2 - 40x + 100$$

$$x^2 - 26x + 25 = 0$$

$x_1 = 25;\quad x_2 = 1$

x_2 entfällt

Aufgabe 2

a) $\lim\limits_{x\to 0}\frac{e^{\sin 2x} - e^{\sin x}}{x} = \lim\limits_{x\to 0}\frac{e^{\sin 2x}\cdot 2\cos 2x - e^{\sin x}\cdot\cos x}{1} = e^0\cdot 2\cdot 1 - e^0 .1 = 1$

b) $\lim\limits_{n\to\infty} n\cdot\left(\frac{4n-1}{n+3} - 4\right) = \lim\limits_{n\to\infty} n\cdot\left(\frac{4n-1-4n-12}{n+3}\right) = \lim\limits_{n\to\infty}\left(\frac{-13n}{n+3}\right) = -13$

Aufgabe 3

a) $f(x) = x^2 + x$ $\qquad$ $h(x) = \ln(x^2+1)$

$f'(x) = 2x + 1$ $\qquad$ $h'(x) = \frac{2x}{x^2+1}$

$$2x+1 = \frac{2x}{x^2+1}$$

$$(2x+1)(x^2+1) = 2x$$

$$2x^3 + x^2 + 2x + 1 = 2x$$

$$2x^3 + x^2 + 1 = 0$$

$$x_0 = -1$$

Keine weiteren reellen Nullstellen

$$f(-1) = 0$$

$$h(-1) = \ln 2$$

Tangenten: $y_1 = -x - 1$

$y_2 = -x + \ln 2 - 1 = -x - 0{,}306853$

b) $f(x) = \begin{cases} x^2, & x \le 1 \\ ax^2 + bx + c, & x > 1 \end{cases}$

$$f'(x) = \begin{cases} 2x \\ 2ax + b \end{cases}$$

$$f(2) = 0 \Leftrightarrow 4a + 2b + c = 0$$

Stetigkeit:

$$\lim_{x \to 1} x^2 = 1$$

$$\lim_{x \to 1} ax^2 + bx + c = a + b + c$$

$$\Rightarrow a + b + c = 1$$

Differenzierbarkeit:

$$\lim_{x \to 1} 2x = 2$$

$$\lim_{x \to 1} 2ax + b = 2a + b$$

$$\Rightarrow 2a + b = 2$$

$$a = -3;\ b = 8;\ c = -4$$

Aufgabe 4

a) $y = a \ln x + bx^2 + x$

$$y' = \frac{a}{x} + 2bx + 1$$

$$y'(1) = 0 \Leftrightarrow a + 2b + 1 = 0$$

$$y'(2) = 0 \Leftrightarrow \frac{a}{2} + 4b + 1 = 0$$

$$\Rightarrow a = -\frac{2}{3};\ b = -\frac{1}{6}$$

$$y'' = -\frac{a}{x^2} + 2b = \frac{2}{3x^2} - \frac{1}{3}$$

$$y''(1) = \frac{1}{3} > 0 \Rightarrow Min.$$

$$y''(2) = -\frac{1}{6} < 0 \Rightarrow Max.$$

b) $f(x) = \sqrt[3]{9 - x^2}, -3 \le x \le 3$

Fläche: $2x \cdot \sqrt[3]{9 - x^2}$ maximieren!

$$z' = 2 \cdot \sqrt[3]{9 - x^2} + 2x \cdot \frac{1}{3} \cdot \left(9 - x^2\right)^{-2/3} \cdot (-2x)$$

$$= 2 \cdot \sqrt[3]{9 - x^2} - \frac{4}{3} x^2 \frac{1}{\sqrt[3]{\left(9 - x^2\right)^2}}$$

$$z' = 0 \Leftrightarrow 2 \cdot \left(9 - x^2\right) = \frac{4}{3} x^2 \Leftrightarrow \frac{10}{3} x^2 = 18 \Leftrightarrow x^2 = 5{,}4 \Leftrightarrow x = \pm 2{,}32379$$

Länge: 4,65 Breite: 1,53

Aufgabe 5

a)

$$\int \frac{e^{2x}dx}{\sqrt[4]{e^x+1}} \qquad \text{Substitution:} \quad e^x+1=z, \qquad dz=e^x dx$$

$$=\int \frac{e^x e^x dx}{\sqrt[4]{e^x+1}}=\int \frac{z-1}{\sqrt[4]{z}}dz=\int \frac{z}{\sqrt[4]{z}}dz-\int \frac{1}{\sqrt[4]{z}}dz$$

$$=\frac{4}{7}z^{7/4}-\frac{4}{3}z^{3/4}+C$$

$$=\frac{12}{21}\left(e^x+1\right)^{7/4}-\frac{28}{21}\left(e^x+1\right)^{3/4}$$

$$=\frac{4}{21}\left(e^x+1\right)^{3/4}\left[3\left(e^x+1\right)-7\right]$$

$$=\frac{4}{21}\left(e^x+1\right)^{3/4}\left(3e^x-4\right)$$

$$\int \frac{1-\tan x}{1+\tan x}dx=\int \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}dx=\int \frac{\cos x-\sin x}{\cos x+\sin x}dx=\ln\left|\cos x+\sin x\right|+C$$

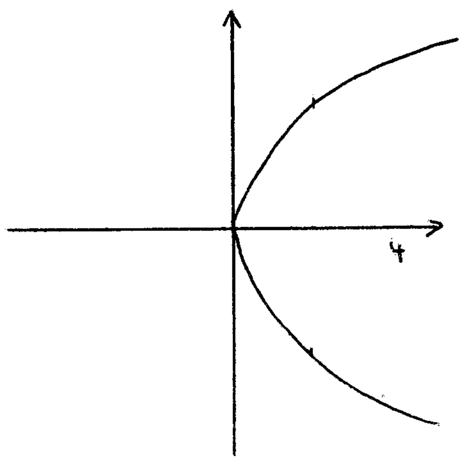
b) $x^2+y^2=16, \qquad y^2=6x$

$16-x^2=6x$

$x^2+6x-16=0$

$x_1=2;\ x_2=-8$

x_2 ist außerhalb des Definitionsbereichs



$$\int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16-x^2}\,dx = \sqrt{6}\frac{2}{3}x^{3/2}\Big|_0^2 + \frac{1}{2}\left(x\sqrt{16-x^2} + 16\arcsin\frac{x}{4}\right)\Big|_2^4$$

$$= \frac{2}{3}\sqrt{6}\cdot 2^{3/2} + \frac{1}{2}\left(4\sqrt{0} + 16\arcsin 1 - 2\sqrt{12} - 16\arcsin\frac{2}{4}\right)$$

$$= \frac{2}{3}\sqrt{48} + \frac{1}{2}\left(16\frac{\pi}{2} - 2\sqrt{12} - 16\frac{\pi}{6}\right)$$

$$= \frac{8}{3}\sqrt{3} + \frac{1}{2}\left(8\pi - 4\sqrt{3} - \frac{8}{3}\pi\right) = \frac{2}{3}\sqrt{3} + \frac{8}{3}\pi \approx 9{,}532281$$

Flächeninhalt mal 2 ergibt $\frac{4}{3}\left(\sqrt{3} + 4\pi\right) = 19{,}065$

2. Fläche: Kreis – Flächeninhalt 1: $16\pi - \frac{4}{3}\left(\sqrt{3} + 4\pi\right) = \frac{4}{3}\left(8\pi - \sqrt{3}\right) = 31{,}2$

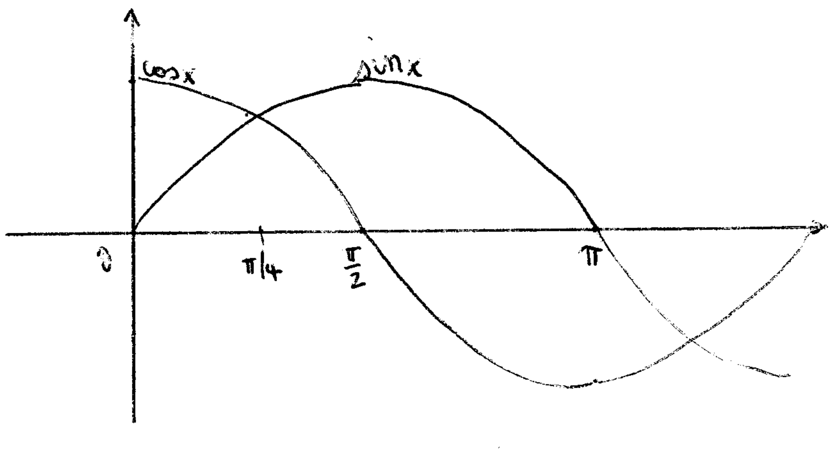
c)

$$\int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx + \int_{\pi/2}^{\pi} \sin x dx = \cos x + \sin x\Big|_{\pi/4}^{\pi/2} + (-\cos x)\Big|_{\pi/2}^{\pi}$$

$$= 1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + 1 = 2 - \sqrt{2}$$

oder

$$\int_0^{\pi/4} \sin x dx + \int_{\pi/4}^{\pi/2} \cos x dx = -\cos x\Big|_0^{\pi/4} + \sin x\Big|_{\pi/4}^{\pi/2} = -\frac{\sqrt{2}}{2} + 1 + 1 - \frac{\sqrt{2}}{2} = 2 - \sqrt{2}$$



Aufgabe 6

a) $z = \left(i \cdot e^{i\pi}\right)^2 = i^2 \cdot e^{2\pi i} = -(\cos 2\pi + i \sin 2\pi) = -1$

Re $z = -1$; Im $z = 0$

b) $z^5 + 4z^3 + 3z = 0$

$z_1 = 0$

$z^4 + 4z^2 + 3 = 0$

$z^2 = -2 \pm \sqrt{4-3} = -2 \pm \sqrt{1} = \begin{cases} -3 \\ -1 \end{cases}$

$-3 \Rightarrow z_1 = \sqrt{3}i;\ z_2 = -\sqrt{3}i$

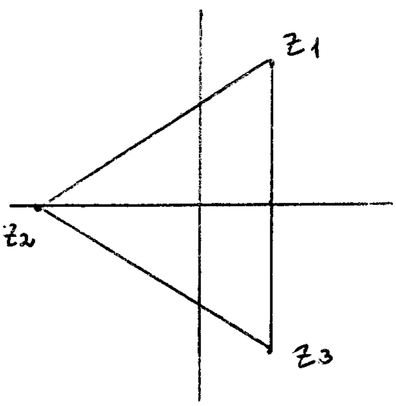
$-1 \Rightarrow z_3 = i;\ z_4 = -i$

c) $\sqrt[3]{-27} = \sqrt[3]{27 cis \pi}$

$z_1 = 3 \cdot (\cos 60° + i \sin 60°) = 3 \cdot \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = \frac{3}{2} + i\frac{3}{2}\sqrt{3}$

$z_2 = 3 \cdot (\cos 180° + i \sin 180°) = 3 \cdot (-1 + i \cdot 0) = -3$

$z_3 = 3 \cdot (\cos 300° + i \sin 300°) = \frac{3}{2} - i\frac{3}{2}\sqrt{3}$



d) $|z-3| = |z-i|$

$|a + ib - 3| = |a + ib - i|$

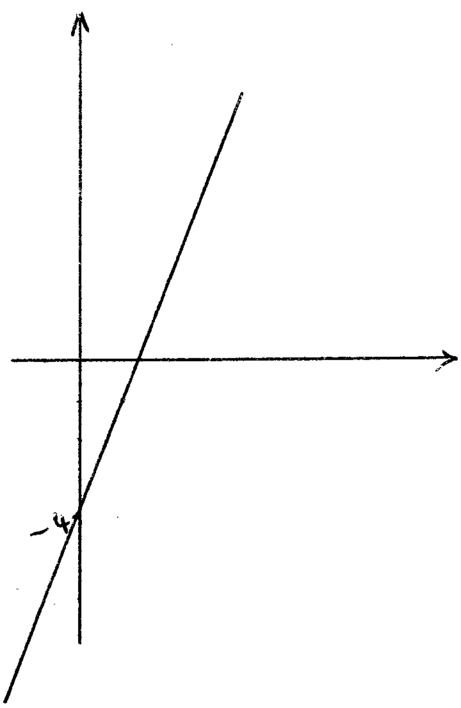
$\sqrt{(a-3)^2 + b^2} = \sqrt{a^2 + (b-1)^2}$

$a^2 - 6a + 9 + b^2 = a^2 + b^2 - 2b + 1$

$2b = 6a - 9 + 1$

$b = 3a - 4$

Gerade



Aufgabe 7

a) $\vec{p}$ parallel zu $\vec{q}$: $\vec{p} = \lambda\vec{q}$

$$2 = 3\lambda \Rightarrow \lambda = \frac{2}{3}$$

$$k = 5\lambda \Rightarrow k = 5 \cdot \frac{2}{3} = \frac{10}{3}$$

$\vec{p} \perp \vec{q}$:

$$6 + 5k = 0$$

$$5k = -6$$

$$k = -\frac{6}{5}$$

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{6+5k}{\sqrt{4+k^2}\sqrt{9+25}} = \frac{6+5k}{\sqrt{4+k^2}\sqrt{34}}$$

$$\sqrt{2}\sqrt{4+k^2}\sqrt{34} = 2 \cdot (6+5k)$$

$$2 \cdot (4+k^2) \cdot 34 = 4 \cdot (36 + 60k + 25k^2)$$

$$32k^2 + 240k - 128 = 0$$

$$k^2 + 7{,}5k - 4 = 0$$

$$k_1 = 0{,}5; \;\; k_2 = -8$$

b) $\vec{a} \times \vec{b} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix}$

$$\left| \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix} \right| = \sqrt{16+1+9} = \sqrt{26}$$

Fläche: $\dfrac{\sqrt{26}}{2}$

Höhe: $\dfrac{\sqrt{26}}{\sqrt{1+4+4}} = \dfrac{\sqrt{26}}{3}$